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Robustness Analysis and Synthesis for Uncertain Nonlinear Systems

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Abstract

The stability and performance robustness analysis for a class of uncertain nonlinear systems with bounded structured uncertainties are characterized in terms of various types of **nonlinear matrix inequalities** (NLMIs). As in the linear case, scalings or multipliers are used to find Lyapunov functions that give sufficient conditions, and the resulting NLMIs yield convex feasibility problem. For these problems, robustness analysis is essentially no harder than stability analysis of the system with no uncertainty. Sufficient conditions for the solvability of related robust synthesis problems are developed in terms of NLMIs as well.

1 Introduction

In this paper, the robustness analysis and synthesis problems for a class of nonlinear systems subject to bounded structured dynamic uncertainties are addressed. The robustness analysis is to determine that under what conditions for the nominal system, the uncertain system is stable and/or satisfies some performance for all admissible uncertainty, while the robustness synthesis problem is to decide under what conditions there are feedback control laws for the uncertain systems such that the closed loop uncertain systems have the required robustness, and then design the control law.

There have been a lot of research activities in the robustness analysis and synthesis since the small-gain theorem was introduced by Zames and Sandberg in the 1960's [5, 4, 6, 18, 15, 17]. These characterizations about the analysis of robust stability and performance for uncertain systems, which are treated in the input/output setting, are essentially reduced to system gain analysis. For linear systems, the $\mathcal{L}_2$-stability and $\mathcal{H}_\infty$-performance robustness is further characterized in state-space as LMIs by the use of KYP lemma [13, 16, 9, 15, 1, 3].

In this paper, we give state-space characterizations of stability and performance robustness for uncertain nonlinear systems, and consider both analysis and synthesis problems. By robust stability, we mean that the feedback system is asymptotically stable for each admissible uncertainty; the robust performance means that the uncertain system is asymptotically stable and has $\mathcal{L}_2$-gain ≤ 1. The treatments of the robustness issues in this paper are motivated by the small gain theorem and its recent extensions, together with the LMI characterization of results in the linear case. As in the linear case, scalings or multipliers are used to find Lyapunov functions. All of the conditions are characterized in terms of **nonlinear matrix inequalities** (NLMIs), which are natural generalizations of the linear matrix inequalities (LMIs) that appear in linear robustness analysis and synthesis.

The NLMI characterizations offer certain potentially attractive computational features. In particular, like the linear case, the NLMIs trivially give convex conditions on the unknowns and essentially make robustness analysis computation no more difficult than stability analysis with no uncertainty. Unfortunately, unlike the linear case, the NLMI conditions involve neither a finite number of unknowns nor a finite number of constraints, as would be expected from consideration of computation of Lyapunov function for nonlinear systems. Thus the computational advantages of NLMIs are far less immediate than for LMIs. Clearly, much additional work will be needed on the computational aspects, by exploring special classes of nonlinear systems or by developing finite-dimensional approximations to these infinite dimensional problems.

The rest of the paper is organized as follows. In section 2, some (NLMI) results about $\mathcal{L}_2$-gain analysis for nonlinear systems are reviewed. In section 3, the stability robustness with structured uncertainty is characterized. In section 4, the robust performance analysis is conducted. In section 5, the robustness synthesis problem is discussed; just the state feedback performance robustness synthesis problem is treated. The characterizations of both robustness analysis and robustness synthesis are in terms of NLMIs. Some computational issues are addressed in section 6.

2 Preliminaries: $\mathcal{L}_2$-Gains

Consider the following input-affine nonlinear systems,

$$G: \begin{cases} \dot{x} = f(x) + g(x)w \\ z = h(x) + k(x)w \end{cases} \tag{2.1}$$

where $x \in \mathbf{R}^n$ is state vector, $w \in \mathbf{R}^p$ and $z \in \mathbf{R}^q$ are input and output vectors, respectively. It is assumed that $f, g, h, k \in \mathbf{C}^0$ are verctor or matrix valued function, and $f(0) = 0, h(0) = 0$. It is assumed that the system evolves on a convex open subset $\mathbf{X} \subset \mathbf{R}^n$ containing the origin. Thus, $0 \in \mathbf{R}^n$ is the equilibrium of the system with $w = 0$.

Definition 2.1 *The system (2.1) with initial state $x(0) = 0$ is said to have $\mathcal{L}_2$-gain less than or equal to γ for some $\gamma > 0$ if*

$$\int_0^T \|z(t)\|^2\, dt \leq \gamma^2 \int_0^T \|w(t)\|^2\, dt \tag{2.2}$$

for all $T \geq 0$ and $w(t) \in \mathcal{L}_2^e(\mathbf{R}^+)$.

The following theorem from [11] characterizes the $\mathcal{L}_2$-gains of a class of stable nonlinear systems.

Theorem 2.2 *Consider system (2.1)) with $R(x) = I - k^T(x)k(x) > 0$, it is asymptotically stable and has $\mathcal{L}_2$-gain ≤ 1, if there exist a $\mathbf{C}^1$ positive definite function $V : \mathbf{X} \rightarrow \mathbf{R}^+$ such that*

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)h(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) + h^T(x)k(x) \\ \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x) + k^T(x)h(x) & k^T(x)k(x) - I \end{bmatrix} < 0, \tag{2.3}$$

for all $x \in \mathbf{X} \setminus \{0\}$.

Remark 2.3 *Note that the condition in Theorem 2.2 are affine in $V(x)$, and all solutions form convex sets. In the special case where g and k are identically zero, (2.3) reduces to the standard stability test with Lyapunov function V. These inequalities are*

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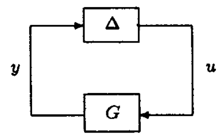
actually differential linear (or affine) matrix inequalities, but we will refer to them as **nonlinear matrix inequalities** *(NLMIs) to emphasize their use in nonlinear problems. All of the conditions that are derived for the analysis problems in the remainder of this paper are similarly convex, and this property will not be discussed for each problem.*

3 Stability Robustness Analysis

Consider the following uncertain system which has a feedback structure,



where the nominal system G has an input-affine realization:

$$G : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) + k(x)u \end{cases} \tag{3.1}$$

with $f, g, h, k \in \mathbf{C}^0$ and $f(0) = 0, h(0) = 0$, the uncertainty Δ belongs to a norm-bounded structured set:

$$\begin{aligned} \mathbf{B\Delta} := \{\Delta = \mathrm{DIAG}\{\Delta_1, \Delta_2, \cdots, \Delta_N\} : \\ \Delta \text{ is causal stable and has } \mathcal{L}_2\text{-gain} \leq 1\}. \end{aligned} \tag{3.2}$$

Note that $\Delta := \mathrm{DIAG}\{\Delta_1, \Delta_2, \cdots, \Delta_N\} \in \mathbf{B\Delta}$ if and only if block Δ_i, which is a causal stable nonlinear system, has $\mathcal{L}_2$-gain ≤ 1. Moreover, the following stronger assumption is made.

Assumption 3.1 *For each $\Delta := \{\Delta = \mathrm{DIAG}\{\Delta_1, \Delta_2, \cdots, \Delta_N\} \in \mathbf{B\Delta}$, Δ_i ($i \in \{1, 2, \cdots, N\}$) has the following realization:*

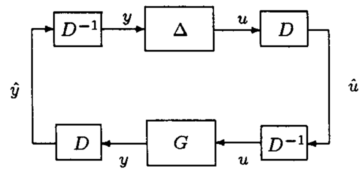
$$\begin{cases} \dot{\xi}_i = f_i(\xi_i, t, y_i) \\ u_i = h_i(\xi_i, t, y_i) \end{cases}$$

which evolves on $\mathbf{X}_i$ and $\dot{\xi}_i = f_i(\xi_i, t, 0)$ is asymptotically stable around $0 \in \mathbf{X}_i$; in addition, there is a $\mathbf{C}^1$ positive definite function U_i such that $\dot{U}_i(\xi_i) \leq \|y_i\|^2 - \|u_i\|^2$.

Definition 3.2 *The uncertain system is* **robustly stable** *if for each $\Delta \in \mathbf{B\Delta}$, the feedback system is well-posed and asymptotically stable around 0.*

For simplicity, it is assumed that $k(x) = 0$ in (3.1), in which case, each uncertainty $\Delta \in \mathbf{\Delta}$ is fed back to the nominal system G in the well-posed manner.

To reduce possible conservatism arising from the uncertainty structure, we next perform a standard scaling manipulation [4] as shown in the following diagram,



where D is some real invertible matrix. Note that the above uncertain system is equivalent to the one in the original diagram. Define a matrix set $\mathcal{D}$ as

$$\mathcal{D} := \{\mathrm{DIAG}\{d_1 I, d_2 I, \cdots, d_N I\} : d_i \in \mathbf{R}, d_i > 0\} \tag{3.3}$$

where each of the identity matrices is compatible in dimension with the corresponding uncertainty Δ_i. It is noted that for each $D \in \mathcal{D}$, $\Delta \in \mathbf{B\Delta}$ if and only if $D\Delta D^{-1} \in \mathbf{B\Delta}$, since the $\mathcal{L}_2$-gains of Δ and $D\Delta D^{-1}$ are the same. Therefore, $D\Delta D^{-1}$ is a legal (transformed) uncertainty structure; it satisfies Assumption 3.1 as does Δ. Thus, we may consider the scaled system DGD^{-1}, in stead of the original one G, to reduce the possible conservatism arising from the uncertainty structure; this scaling treatment was also justified in [18, 17]. This consideration is reflected in the following theorem about the robust stability for uncertain nonlinear systems.

Theorem 3.3 *Consider the uncertain system with nominal system as in (3.1) with $k(x) = 0$ and the admissible uncertainty set (3.2) under Assumption 3.1, it is robustly stable if there exist a positive definite function $V : \mathbf{X} \to \mathbf{R}^+$ and a positive definite matrix $Q \in \mathcal{D}$ such that the following NLMI holds:*

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x) f(x) + h^T(x) Q h(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x) g(x) \\ \frac{1}{2} g^T(x) \frac{\partial V^T}{\partial x}(x) & -Q \end{bmatrix} < 0 \tag{3.4}$$

for all $x \in \mathbf{X} \setminus \{0\}$.

Proof

Define $\hat{g}(x) = g(x) Q^{-1/2}$ and $\hat{h}(x) = Q^{1/2} h(x)$. By Schur complements argument, the above inequality (3.4) is equivalent to the following HJI:

$$\hat{\mathcal{H}}(x) := \frac{\partial V}{\partial x}(x) f(x) + \hat{h}^T(x)\hat{h}(x) + \frac{1}{4}\frac{\partial V}{\partial x}(x)\hat{g}(x)\hat{g}^T(x)\frac{\partial V^T}{\partial x}(x) < 0. \tag{3.5}$$

for all $x \in \mathbf{X} \setminus \{0\}$. Take V as defined in the statement, and define $\hat{u} = Q^{1/2} u$ and $\hat{y} = Q^{1/2} y$, then

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x}(x)(f(x) + \hat{g}(x)\hat{u}) \\ &= \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2 - \left\| \hat{u}(t) + \frac{1}{2}\hat{g}^T(x)\frac{\partial V^T}{\partial x}(x) \right\|^2 + \hat{\mathcal{H}}(x). \\ &\leq \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2 + \hat{\mathcal{H}}(x). \end{aligned} \tag{3.6}$$

Denote $Q^{1/2} := \mathrm{DIAG}\{q_1 I, q_2 I, \cdots, q_N I\} \in \mathcal{D}$. From the Assumption 3.1, for each $\Delta \in \mathbf{B\Delta}$, there is a positive definite function $U_i : \mathbf{X}_i \to \mathbf{R}^+$ for nonlinear system $q_i \Delta_i q_i^{-1}$ for each $i \in \{1, 2, \cdots, N\}$ such that

$$\dot{U}_i(\xi_i) \leq \|\hat{y}_i(t)\|^2 - \|\hat{u}_i(t)\|^2 \tag{3.7}$$

where $\hat{u}_i = q_i u_i$, $\hat{y}_i = q_i y_i$ and ξ_i is the state vector of Δ_i on $\mathbf{X}_i$. Therefore, $\|\hat{u}\|^2 = \sum_{i=1}^N \|\hat{u}_i\|^2$, $\|\hat{y}\|^2 = \sum_{i=1}^N \|\hat{y}_i\|^2$.

Define a positive definite function W on $\mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N$ as

$$W(x, \xi_1, \cdots, \xi_N) = V(x) + \sum_{i=1}^N U_i(\xi_i) \tag{3.8}$$

So from (3.6) and (3.7), it follows that

$$\begin{aligned} \dot{W}(x, \xi_1, \cdots, \xi_N) &\leq \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2 + \hat{\mathcal{H}}(x) + \sum_{i=1}^N (\|\hat{y}_i(t)\|^2 - \|\hat{u}_i(t)\|^2) \\ &\leq \hat{\mathcal{H}}(x) \end{aligned}$$

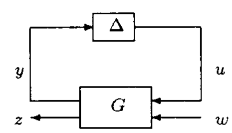
Hence, if $\dot{W}(x, \xi_1, \cdots, \xi_N) = 0$, then $\mathcal{H}(x) = 0$; it in turn implies $x = 0$. On the other hand, $x = 0$ implies $y = 0$, so the feedback system evolves on the following set,

$$\begin{aligned} \{(x, \xi_1, \cdots, \xi_N) \in \mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N : \\ x = 0, \dot{\xi}_i = f_i(\xi_i, t, 0) \forall i \in \{1, 2, \cdots, N\}\}. \end{aligned}$$

But by the Assumption 3.1, $\dot{\xi}_i = f_i(\xi_i, t, 0)$ implies $\xi_i(t) \to 0$ as $t \to \infty$. By LaSalle's theorem, $W : \mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N \to \mathbf{R}^+$ is a Lyapunov function for the feedback system, and the system is asymptotically stable. Therefore, the uncertain system is robustly stable. □

4 Performance Robustness Analysis

Consider the following feedback uncertain system.



where w is some external disturbance vector, and it is assumed $w \in \mathcal{L}_2^e(\mathbf{R}^+)$; z is the regulated signal vector. The nominal plant G has the following input affine realization

$$G : \begin{cases} \dot{x} = f(x) + g_1(x)u + g_2(x)w \\ y = h_1(x) + k_{11}(x)u + k_{12}(x)w \\ z = h_2(x) + k_{21}(x)u + k_{22}(x)w \end{cases} \tag{4.1}$$

with $f, g_i, h_j, k_{ij} \in \mathbf{C}^0$ and $f(0) = 0, h_j(0) = 0$. It is assumed that $k_{11}(x) = 0$ to guarantee the well-posedness of the feedback structure; the admissible uncertainty structure is described by the bounded structured set $\mathbf{B\Delta}$ defined in (3.2), the scaling matrix set $\mathcal{D}$ is defined in (3.3).

Definition 4.1 *The uncertain system depicted above satisfies* **robust performance** *if for each $\Delta \in \mathbf{B\Delta}$, the corresponding feedback system has $\mathcal{L}_2$-gain ≤ 1, and is asymptotically stable around 0 for $w = 0$.*

In this section, we will examine under what conditions, the uncertain system depicted above has robust performance. Define

$$g(x) := \begin{bmatrix} g_1(x) & g_2(x) \end{bmatrix},$$

$$h(x) := \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix}, \quad k(x) := \begin{bmatrix} k_{11}(x) & k_{12}(x) \\ k_{21}(x) & k_{22}(x) \end{bmatrix}.$$

We first have the following result about robust performance analysis, in which the scaling manipulation is conducted as did in the last section.

Theorem 4.2 *Consider the uncertain system with nominal plant (4.1) with $k_{11}(x) = 0$ and uncertainty set $\mathbf{B\Delta}$ (3.2) under Assumption 3.1, then it has robust performance if there exist a positive definite $\mathbf{C}^1$ function $V : \mathbf{X} \to \mathbf{R}$ and a positive definite matrix $\hat{Q} \in \mathcal{D}$ such that the following NLMI holds:*

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)Qh(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) + h^T(x)Qk(x) \\ \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x) + k^T(x)Qh(x) & k^T(x)Qk(x) - Q \end{bmatrix} < 0 \tag{4.2}$$

with $Q = \begin{bmatrix} \hat{Q} & 0 \\ 0 & I \end{bmatrix}$ for all $x \in \mathbf{X} \setminus \{0\}$.

Proof

Define $\hat{g}(x) := g(x)Q^{-1/2}$, $\hat{h}(x) := Q^{1/2}h(x)$ and $\hat{k}(x) := Q^{1/2}k(x)Q^{-1/2}$. Using Schur complements argument and the inequality (4.2) is equivalent to the following two inequalities.

$$\hat{R}(x) := I - \hat{k}^T(x)\hat{k}(x) > 0,$$

and

$$\hat{\mathcal{H}}(x) := \frac{\partial V}{\partial x}(x)f(x) + \hat{h}^T(x)\hat{h}(x) +$$
$$+ (\frac{1}{2}\frac{\partial V}{\partial x}(x)\hat{g}(x) + \hat{h}^T(x)\hat{k}(x))(I - \hat{k}^T(x)\hat{k}(x))^{-1}(x) \cdot$$
$$\cdot (\frac{1}{2}\hat{g}^T(x)\frac{\partial V^T}{\partial x}(x) + \hat{k}^T(x)\hat{h}(x)) < 0. \tag{4.3}$$

for all $x \in \mathbf{X} \setminus \{0\}$. Take V as defined in the statement; define $\hat{u} := Q^{1/2}\begin{bmatrix} u \\ w \end{bmatrix}$ and $\hat{y} = Q^{1/2}\begin{bmatrix} y \\ z \end{bmatrix}$ then

$$\dot{V}(x) \leq \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2 + \hat{\mathcal{H}}(x). \tag{4.4}$$

Denote $\hat{Q}^{1/2} = \mathrm{DIAG}\{q_1 I, q_2 I, \cdots, q_N I\}$. From the Assumption 3.1, for all $\Delta \in \mathbf{B\Delta}$, there is a positive definite function $U_i : \mathbf{X}_i \to \mathbf{R}^+$ for nonlinear system $q_i \Delta_i q_i^{-1}$ for each $i \in \{1, 2, \cdots, N\}$ such that

$$\dot{U}_i(\xi_i) \leq \|\hat{y}_i(t)\|^2 - \|\hat{u}_i(t)\|^2 \tag{4.5}$$

where $\hat{u}_i = q_i u_i$, $\hat{y}_i = q_i y_i$ and ξ_i is the state vector of Δ_i on $\mathbf{X}_i$. Therefore,

$$\|\hat{u}\|^2 = \sum_{i=1}^{N} \|\hat{u}_i\|^2 + \|w\|^2, \quad \|\hat{y}\|^2 = \sum_{i=1}^{N} \|\hat{y}_i\|^2 + \|z\|^2.$$

Define a positive definite function W on $\mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N$ as

$$W(x, \xi_1, \cdots, \xi_N) = V(x) + \sum_{i=1}^{N} U_i(\xi_i) \tag{4.6}$$

So from (4.4) and (4.5), it follows that

$$\dot{W}(x, \xi_1, \cdots, \xi_N) \leq \|\hat{u}(t)\|^2 - \|\hat{y}(t)\|^2 + \hat{\mathcal{H}}(x) + \sum_{i=1}^{N}(\|\hat{y}_i(t)\|^2 - \|\hat{u}_i(t)\|^2)$$
$$\leq \|w\|^2 - \|z\|^2 + \hat{\mathcal{H}}(V, Q, x) \leq \|w\|^2 - \|z\|^2 \tag{4.7}$$

The latter inequality implies

$$\int_0^T \|z\|^2 \, dt \leq \int_0^T \|w\|^2 \, dt \tag{4.8}$$

for all $T \in \mathbf{R}^+$, i.e., the feedback system has $\mathcal{L}_2$-gain ≤ 1.

Next, we consider the asymptotic stability for $w = 0$. In this case, (4.7) becomes

$$\dot{W}(x, \xi_1, \cdots, \xi_N) \leq -\|z\|^2 + \hat{\mathcal{H}}(x).$$

Thence, if $\dot{W}(x, \xi_1, \cdots, \xi_N) = 0$, then $\hat{\mathcal{H}}(x) = 0$, it in turn implies $x = 0$ by (4.3). But $x = 0$ implies $y = 0$, therefore $\xi_i(t) \to 0$ as $t \to \infty$, for $\dot{\xi}_i = f_i(\xi_i, t, 0)$. By LaSalle's theorem, $W : \mathbf{X} \times \mathbf{X}_1 \times \cdots \times \mathbf{X}_N \to \mathbf{R}^+$ is a Lyapunov function for the given closed loop system, and the system is asymptotically stable.

Therefore, we conclude that the uncertain system is of robust performance. □

Next, we further relax the condition for the last theorem to get an alternative characterization for the robust performance of the depicted uncertain system with nominal system (4.1).

Assumption 4.3 *Consider the nominal system (4.1), define a new system*

$$\begin{cases} \dot{x} = f(x) + g_1(x)u \\ z = h_2(x) + k_{21}(x)u \end{cases}$$

The solution for all possible $u(t)$ under the constraint $z(t) = 0$ satisfies $x(t) = 0$ for all $t \in \mathbf{R}^+$.

It is noted that in the linear case, the above assumption corresponds to the condition that the system has no transmission zero.

Theorem 4.4 *Under assumptions 4.3 and 3.1, the uncertain system (4.1) has robust performance if there exist a positive definite $\mathbf{C}^1$ function $V : \mathbf{X} \to \mathbf{R}$ and a positive definite matrix $\hat{Q} \in \mathcal{D}$ such that the following NLMIs hold:*

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)Qh(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) + h^T(x)Qk(x) \\ \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x) + k^T(x)Qh(x) & k^T(x)Qk(x) - Q \end{bmatrix} \leq 0$$

$$k^T(x)Qk(x) - Q < 0$$

with $g(x), h(x), k(x)$ defined previously and $Q := \begin{bmatrix} \hat{Q} & 0 \\ 0 & I \end{bmatrix}$ for all $x \in \mathbf{X}$.

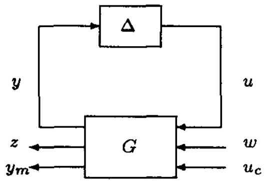
Proof

The proof is given in [12]. □

5 Robustness Synthesis of Uncertain Systems

In the last two sections, the robustness of uncertain systems is essentially characterized as some small $\mathcal{L}_2$-gain conditions for (scaled) nominal systems modulo some appropriate stabilizing conditions. So the robustness synthesis can be pursued by combining the robustness analysis results in the last two sections with the treatments of $\mathcal{H}_\infty$-control synthesis (see for example [19, 2, 10]). We just take the performances robustness synthesis problem as an example, the robust stabilization problem can be done similarly. Technically, we closely follow the treatments in [19, 10, 11]. Just the state feedback solutions are provided; the output feedback case can be done similarly by just modifying the treatments in [2, 10, 11]. The robust stabilization with unstructured uncertainty is considered in [20].

Consider an uncertain system which has the following feedback structure.



where $w \in \mathcal{L}_2^e(\mathbf{R}^+)$ is some external disturbance vector, z is the regulated signal vector, y_m is the measured output vector, and based on which the control input vector u_c is produced. The nominal plant G has the following realization

$$G: \begin{cases} \dot{x} = f(x) + g_1(x)u + g_2(x)w + g_3(x)u_c \\ y = h_1(x) + k_{11}(x)u + k_{12}(x)w + k_{13}(x)u_c \\ z = h_2(x) + k_{21}(x)u + k_{22}(x)w + k_{23}(x)u_c \\ y_m = h_3(x) + k_{31}(x)u + k_{32}(x)w + k_{33}(x)u_c \end{cases} \quad (5.1)$$

where $f, g_i, h_j, k_{ij} \in \mathbf{C}^0$, and $f(0) = 0, h_j(0) = 0$, for $i, j = 1, 2, 3$. It is assumed $k_{11}(x) = 0$ to insure the well-posedness of the feedback structure. In this section, the state vector x of the nominal system is directly measured, i.e., $y_m = x$; the admissible uncertainty structure is a bounded structured set $\mathbf{B\Delta}$ defined in (3.2). The scaling matrix set $\mathcal{D}$ is defined in (3.3).

The performance robustness synthesis problem by state feedback is defined as follows.

Definition 5.1 (State Feedback Synthesis Problem) *Find a state feedback law $u_c = K(x)$ with $K \in \mathbf{C}^0$ and $K(0) = 0$ for the uncertain system depicted above such that the closed loop uncertain system satisfies robust performance.*

Next, we consider two cases about robustness synthesis by state feedback. Basically, the characterization follows from a two-step treatment: (i) Characterize the $\mathcal{L}_2$-gain of the state feedback system by theorem 4.2 in terms of an NLMI, which depends on the state-feedback; (ii) characterize the NLMI without the feedback.

5.1 State Feedback Solutions

Consider the uncertain system with the nominal plant as (5.1). Define

$$g(x) := \begin{bmatrix} g_1(x) & g_2(x) \end{bmatrix}, h(x) := \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix},$$

$$k_1(x) := \begin{bmatrix} k_{11}(x) & k_{12}(x) \\ k_{21}(x) & k_{22}(x) \end{bmatrix}, k_2(x) := \begin{bmatrix} k_{13}(x) \\ k_{23}(x) \end{bmatrix}.$$

The following structural constraints are imposed for simplicity.

Assumption 5.2 $k_1(x) = 0$, $k_2^T(x)\begin{bmatrix} h(x) & k_2(x) \end{bmatrix} = \begin{bmatrix} 0 & R_0(x) \end{bmatrix}$ *where $R_0(x) > 0$ for all $x \in \mathbf{X}$.*

We first have the following lemma.

Lemma 5.3 *Consider the system defined in (5.1) with the structural Assumption 5.2. The following two statement are equivalent.*

(i) There exist a $\mathbf{C}^0$ vector-valued function $K(x)$ on $\mathbf{X}$, a $\mathbf{C}^1$ positive definite function $V : \mathbf{X} \rightarrow \mathbf{R}^+$, and a positive definite matrix $\hat{Q} \in \mathcal{D}$ such that the following NLMI holds,

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x) f_K(x) + h_K^T(x) Q h_K(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x) g(x) \\ \frac{1}{2} g^T(x) \frac{\partial V^T}{\partial x}(x) & -Q \end{bmatrix} < 0 \quad (5.2)$$

for all $x \in \mathbf{X} \setminus \{0\}$, where

$$f_K(x) = f(x) + g_3(x)K(x), \quad h_K(x) = h(x) + k_2(x)K(x)$$

(ii) There exist a $\mathbf{C}^1$ positive definite function $V : \mathbf{X} \rightarrow \mathbf{R}^+$ and a positive definite matrix $\hat{Q} \in \mathcal{D}$ such that the following hamilton-Jacobi inequality (HJI) holds,

$$\frac{\partial V}{\partial x}(x) f(x) + \frac{1}{4}\frac{\partial V}{\partial x}(x)(g(x)Q^{-1}g^T(x) - g_3(x)R_0^{-1}(x)g_3^T(x)) \cdot \\ \cdot \frac{\partial V^T}{\partial x}(x) + h^T(x)Qh(x) < 0. \quad (5.3)$$

for all $x \in \mathbf{X} \setminus \{0\}$.

Moreover, if (ii) is true, then a state feedback function $K(x)$ makes (i) true is as follows.

$$K(x) = -\frac{1}{2}R_0^{-1}(x) g_3^T(x) \frac{\partial V}{\partial x}(x).$$

Proof

Note that the NLMI in statement (i) is equivalent to the following Hamilton-Jacobi in equality,

$$\frac{\partial V}{\partial x}(x)(f(x) + g_3(x)K(x)) + \frac{1}{4}\frac{\partial V}{\partial x}(x) g(x) Q^{-1} g^T(x) \frac{\partial V^T}{\partial x}(x) + \\ + (h(x) + k_2(x)K(x))^T Q (h(x) + k_2(x)K(x)) < 0$$

for all $x \in \mathbf{X} \setminus \{0\}$. By the same arguments as in [10, Theorem 4.1], the conclusion follows. □

The main result in this subsection is stated as follows.

Theorem 5.4 *Consider the uncertain system with nominal plant as (5.1). Under assumptions 3.1 and 5.2, the state feedback robust performance synthesis problem has a solution if there exist a positive definite $\mathbf{C}^1$ positive definite function $V : \mathbf{X} \rightarrow \mathbf{R}^+$ and a positive definite matrix $\hat{Q} \in \mathcal{D}$ such that the HJI (5.3) holds for all $x \in \mathbf{X} \setminus \{0\}$. Moreover, if $(V(x), Q)$ is such a pair of solutions, then a state feedback function $K(x)$ makes the closed loop system has a robust performance is $K(x) = -\frac{1}{2}R_0^{-1}(x) g_3^T(x) \frac{\partial V}{\partial x}(x)$.*

Proof

Let $(V(x), Q)$ be as in the theorem. By the preceding lemma, there exist a $\mathbf{C}^0$ matrix valued function $K(x)$ on $\mathbf{X}$ defined as $K(x) = -\frac{1}{2}R_0^{-1}(x) g_3^T(x) \frac{\partial V}{\partial x}(x)$, such that the NLMI (5.2) holds for all $x \in \mathbf{X} \setminus \{0\}$. On the other hand, take $u_c = K(x)$ as a state feedback law, so the closed loop nominal system is as follows.

$$G_F: \begin{cases} \dot{x} = (f(x) + g_3(x)K(x)) + g_1(x)u + g_2(x)w \\ y = (h_1(x) + k_{13}(x)K(x)) \\ z = (h_2(x) + k_{23}(x)K(x)) \end{cases}$$

By theorem 4.2, the closed loop uncertain system satisfies robust performance. □

Note that the above characterization is not convex in general. In the next subsection, we will give a convex characterization which have some promising computational properties.

5.2 A Convex Characterization to State Feedback Solutions

In this section, in stead of the nominal plant (5.1), the following nominal plant is examined.

$$G: \begin{cases} \dot{x} = A(x)x + B_1(x)u + B_2(x)w + B_3(x)u_c \\ y = C_1(x)x + D_{11}(x)u + D_{12}(x)w + D_{13}(x)u_c \\ z = C_2(x)x + D_{21}(x)u + D_{22}(x)w + D_{23}(x)u_c \\ y_m = x \end{cases} \quad (5.4)$$

which is evolved on a bounded open set $\mathbf{X}$ containing the origin, where A, B_i, C_j, D_{ij} are $\mathbf{C}^0$ matrix-valued functions on $\mathbf{X}$, and $D_{11}(x) = 0$. Define

$$B(x) := \begin{bmatrix} B_1(x) & B_2(x) \end{bmatrix}, \tilde{B}(x) := \begin{bmatrix} B_3^T(x) & D_{13}^T(x) & D_{23}^T(x) \end{bmatrix},$$

$$C(x) := \begin{bmatrix} C_1(x) \\ C_2(x) \end{bmatrix}, \quad D(x) := \begin{bmatrix} 0 & D_{12}(x) \\ D_{21}(x) & D_{22}(x) \end{bmatrix},$$

and let $\mathcal{N}(\tilde{B}(x))$ be the distribution on $\mathbf{X}$ which annihilates all of the row vectors of $\tilde{B}(x)$. We first have the following lemma.

Lemma 5.5 *The following two statements are equivalent.*

(i) There exist a $\mathbf{C}^0$ matrix valued function $F(x)$, a positive definite matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n \times n}$, and a positive definite matrix $\hat{Q} \in \mathcal{D}$ such that the following NLMI holds:

$$\begin{bmatrix} A_F^T(x)P(x) + P(x)A_F(x) + C_F^T(x)QC_F(x) & P(x)B(x) + C_F^T(x)QD(x) \\ B^T(x)P(x) + D^T(x)QC_F(x) & D^T(x)QD(x) - Q \end{bmatrix} < 0 \quad (5.5)$$

with $Q := \begin{bmatrix} \hat{Q} & 0 \\ 0 & I \end{bmatrix}$ for all $x \in \mathbf{X}$, where

$$A_F(x) = A(x) + B_3(x)F(x), \quad C_F(x) = \begin{bmatrix} C_1(x) + D_{13}(x)F(x) \\ C_2(x) + D_{23}(x)F(x) \end{bmatrix}.$$

(ii) There exist a positive definite matrix-valued function $X : \mathbf{X} \rightarrow \mathbf{R}^{n\times n}$ and a positive definite matrix $\hat{Y} \in \mathcal{D}$ such that the following NLMI holds:

$$B_{\perp}^T(x) \begin{bmatrix} A(x)X(x) + X(x)A^T(x) + B^T(x)YB(x) & X(x)C^T(x) + B(x)YD^T(x) \\ C(x)X(x) + D(x)YB^T(x) & D(x)YD^T(x) - Y \end{bmatrix} B_{\perp}(x) < 0 \tag{5.6}$$

with $Y := \begin{bmatrix} \hat{Y} & 0 \\ 0 & I \end{bmatrix}$, and $B_{\perp}(x)$ is a $\mathbf{C}^0$ matrix-valued function on $\mathbf{X}$ such that $span(B_{\perp}(x)) = \mathcal{N}(\tilde{B}(x))$ for all $x \in \mathbf{X}$.

Moreover, If any one of the above statments holds, then the solutions of the other NLMI can be chosen such that $P(x) = X^{-1}(x)$ and $Q = Y^{-1}$.

Proof

Use the similar arguments in [11, 1]. □

It is noted that the NLMI (5.6) is affine in unknown $P(x)$ and Q. We have following theorem which gives convex characterizations for robust performance synthesis by state feedback.

Theorem 5.6 *Consider the uncertain system with nominal plant defined as (5.4). Under Assumption 3.1, the state feedback robust performance synthesis problem has a solution if there exist a positive definite matrix-valued function $X : \mathbf{X} \rightarrow \mathbf{R}^{n\times n}$ and a positive definite matrix $\hat{Y} \in \mathcal{D}$ such that the NLMI (5.6) holds for all $x \in \mathbf{X}$, and $\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x)$ for some $\mathbf{C}^1$ function V on $\mathbf{X}$ with $V(0) = 0$.*

Proof

Let $(X(x), Y)$ be as in the theorem. By the preceding lemma, there exist a $\mathbf{C}^0$ matrix valued function $F(x)$ on $\mathbf{X}$, take $(P(x), Q) = (X^{-1}(x), Y^{-1})$, then they satisfy the NLMI (5.5) with some $\mathbf{C}^0$ matrix-valued function F on $\mathbf{X}$. Take $u_c = F(x)x$ as a state feedback law, so the closed loop nominal system is as follows.

$$G_F : \begin{cases} \dot{x} = (A(x) + B_3(x)F(x))x + B_1(x)u + B_2(x)w \\ y = (C_1(x) + D_{13}(x)F(x))x + \qquad D_{12}(x)w \\ z = (C_2(x) + D_{23}(x)F(x))x + D_{21}(x)u + D_{22}(x)w \end{cases}$$

We now claim the closed loop uncertain system satisfies robust performance. In fact, (5.5) implies that

$$\begin{bmatrix} x^T(A_F^T(x)P(x) + P(x)A_F(x) + C_F^T(x)QC_F(x))x & x^T(P(x)B(x) + C_F^T(x)QD(x)) \\ (B^T(x)P(x) + D^T(x)QC_F(x))x & D^T(x)QD(x) - Q \end{bmatrix} < 0$$

Now there exists a $\mathbf{C}^1$ function V on $\mathbf{X}$ such that $\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x)$. Thus V is positive definite [11], The conclusion therefore follows from theorem 4.2. □

6 Computational Issues for Robustness

We address computational issues for robustness analysis and synthesis in this section. From the development of the theory in the last three sections, it is noted that the computation about robustness analysis and synthesis involves solving some NLMIs. To be more concrete, we take the following NLMI with respect to $(V(x), Q)$ (i.e. (3.4) or (4.2)) as an example,

$$\begin{bmatrix} \frac{\partial V}{\partial x}(x)f(x) + h^T(x)Qh(x) & \frac{1}{2}\frac{\partial V}{\partial x}(x)g(x) + h^T(x)Qk(x) \\ \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x) + k^T(x)Qh(x) & k^T(x)Qk(x) - Q \end{bmatrix} \leq 0$$

for all $x \in \mathbf{X}$. The computation procedure therefore consists of the following two steps,

- Find $(p(x), Q)$ for some $\mathbf{C}^0$ vector-valued function $p : \mathbf{X} \rightarrow \mathbf{R}^n$ with $p(0) = 0$ and positive definite matrix Q such that
$$\begin{bmatrix} 2p^T(x)f(x) + h^T(x)Qh(x) & p^T(x)g(x) + h^T(x)Qk(x) \\ g^T(x)p(x) + k^T(x)Qh(x) & k^T(x)Qk(x) - Q \end{bmatrix} \leq 0; \tag{6.1}$$
- Check if there is a Lyapunov function $V : \mathbf{X} \rightarrow \mathbf{R}^+$ such that $\frac{\partial V}{\partial x}(x) = 2p^T(x)$ for $x \in \mathbf{X}$.

Next, we pursue the first issue, the second one is considered in [11]. The technique used here is similar to that in [11]. We first consider the solution of (6.1). Suppose $f(x) = A(x)x, g(x) = B(x)$, $h(x) = C(x)x, k(x) = D(x)$, and $p(x) = P(x)x$ for some $\mathbf{C}^0$ matrix-valued function A, B, C, D, and P, then (6.1) is implied by the following NLMI (which is a more conservative characterization),

$$\begin{bmatrix} A^T(x)P(x) + P^T(x)A(x) + C^T(x)QC(x) & P^T(x)B(x) + C^T(x)QD(x) \\ B^T(x)P(x) + D^T(x)QC(x) & D^T(x)QD(x) - Q \end{bmatrix} \leq 0. \tag{6.2}$$

If $(P(x), Q)$ is a solution to (6.2), then $(P(x)x, Q)$ is a solution to (6.1). In this subsection, we need to find a $\mathbf{C}^0$ matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n\times n}$ and a positive definite matrix Q such that $(P(x), Q)$ satisfies NLMI (6.2).

6.1 Solutions to NLMIs

Under some regularity conditions, the existence of positive definite solution $P : \mathbf{X} \rightarrow \mathbf{R}^{n\times n}$ to an NLMI is justified by using [7, lemma 4] in [11, 12]. In the following, we consider inequality $\mathcal{M}(P, Q, x) < 0$, where $\mathcal{M} : \mathbf{R}^{n\times n} \times \mathbf{R}^{p\times p} \times \mathbf{X} \rightarrow \mathbf{R}^{(n+p)\times(n+p)}$ is continuous and satisfies

$$\mathcal{M}(\sum_{k=1}^{N} \alpha_k P_k, \sum_{k=1}^{N} \alpha_k Q_k, x) = \sum_{k=1}^{N} \alpha_k \mathcal{M}(P_k, Q_k, x) \tag{6.3}$$

for all $\alpha_k \geq 0$ with $\sum_{k=1}^{N} \alpha_k = 1$. Therefore, the NLMI (6.2) belongs to this representation. The following theorem further shows that the solutions of $\mathcal{M}(P, Q, x) < 0$ can be chosen to be continuous in the case of interest if pointwise solutions exist.

Theorem 6.1 *Suppose the NLMI $\mathcal{M}(P, Q, x) < 0$ satisfying (6.3) has a positive definite solution $P(x)$ for each $x \in \mathbf{X}$ and $Q > 0$, then there exists a smooth matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n\times n}$ with $P(x) = P^T(x) > 0$, such that $\mathcal{M}(P(x), Q, x) < 0$ for all $x \in \mathbf{X}$.*

Proof

Consider the given NLMI $\mathcal{M}(P, Q, x) \leq 0$ with $x \in \mathbf{X}$, By assumption, there exist a positive definite matrix $Q \in \mathbf{R}^{p\times p}$, and a positive definite $P_x \in \mathbf{R}^{n\times n}$ for each $x \in \mathbf{X}$ such that

$$\mathcal{M}(P_x, Q, x) < 0.$$

By continuity of M with respect to x, there is a $r_x > 0$ such that for all $x_0 \in \mathbf{N}(x) := \{x_0 : \|x_0 - x\| < r_x\}$,

$$\mathcal{M}(P_x, Q, x_0) < 0. \tag{6.4}$$

On the other hand, $\{\mathbf{N}(x)\}|_{x\in\mathbf{X}}$ is an open covering of $X \subset \mathbf{R}^n$, there is a locally finite open subcovering $\{\mathbf{N}_i\}|_{i\in\mathbf{I}}$ for some index set $\mathbf{I}$ which refines $\{\mathbf{N}(x)\}|_{x\in\mathbf{X}}$. By (6.4), $P_i \in \mathbf{R}^{n\times n}$ is taken to be positive definite for each $i \in \mathbf{I}$ such that $\mathcal{M}(P_i, Q, x) < 0$ for all $x \in \mathbf{N}_i$.

By the standard results of partitions of unity, it is known that there is a smooth partition of unity $\{\psi_i\}|_{i\in\mathbf{I}}$ to $\mathbf{X}$ subordinated to the covering $\{\mathbf{N}_i\}|_{i\in\mathbf{I}}$; i.e., ψ_i is smooth and non-negative with support $\textsc{Supp}(\psi_i) \subset \mathbf{N}_i$ for each $i \in \mathbf{I}$, and

$$\sum_{i\in\mathbf{I}} \psi_i(x) = 1, \forall x \in \mathbf{X}. \tag{6.5}$$

Define a matrix-valued function $P : \mathbf{X} \rightarrow \mathbf{R}^{n\times n}$ as

$$P(x) = \sum_{i\in\mathbf{I}} \psi_i(x)P_i, \forall x \in \mathbf{X}, \tag{6.6}$$

which is positive definite and smooth since it is locally a finite sum of smooth positive definite matrix-valued functions. It follows from (6.5), (6.6) and (6.3) that

$$\mathcal{M}(P(x), Q, x) = \mathcal{M}(\sum_{i\in\mathbf{I}} \psi_i(x)P_i, Q, x) = \sum_{i\in\mathbf{I}} \psi_i(x)\mathcal{M}(P_i, Q, x) < 0$$

The last equality holds since the sum is finite for each $x \in \mathbf{X}$.

Thence, the constructed smooth matrix-valued function $P : \mathbf{X} \to \mathbf{R}^{n \times n}$ in (6.6) is positive definite and is a solution to $\mathcal{M}(P(x), Q, x) < 0$. □

Remark 6.2 *From the above proof, it follows that the solutions to NLMIs can be obtained by considering a sequence of LMI problems which lead to local constant solutions. It is noted that if $\mathbf{X}$ is bounded, the number of the involved LMIs is finite. However, as we mentioned before, the existence of the solutions is not enough to get the required conclusion, additionally a Lyapunov function should be exist as discussed in detail in [11].*

A nice convex property for NLMIs is stated by the following proposition whose proof is easy and omitted here.

Theorem 6.3
The $\mathbf{C}^0$ solutions $(P(x), Q)$ to NLMI $\mathcal{M}(P, Q, x) \le 0$ such that $P : \mathbf{X} \to \mathbf{R}^{n \times n}$ is $\mathbf{C}^0$ positive definite with $\frac{\partial V}{\partial x}(x) = 2x^T P(x)$ for some $\mathbf{C}^1$ function $V : \mathbf{X} \to \mathbf{R}$ and $Q > 0$ form a convex set.

6.2 Further Remarks

The above treatments about robustness analysis and synthesis are in terms of NLMIs, which are pointwise LMIs on state set $\mathbf{X}$, modulo some additional constraints on the solutions. From the proof of Theorem 6.2, we know that if $\mathbf{X}$ is bounded, then we only need to solve a finite number of LMIs to get the solution for the NLMI on $\mathbf{X}$. In the following, we will show how to get a constant solution by the method in [3] if the set $\mathbf{X}$ is small enough.

In the light of the notion of global linearization of nonlinear systems developed by Liu et al [8], the considered nonlinear system can be viewed as a parametric uncertain system, i.e., in the current case, the coefficient matrices in (6.2) are assumed in a convex set:

$$[A(x), B(x), C(x), D(x)] \in Co\{[A_i, B_i, C_i, D_i]|_{i \in \{1,2,\cdots,L\}}\}, \forall x \in \mathbf{X},$$

where Co stands for the **convex hull**. In this case, a constant solution $(P, Q) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{q \times q}$ to (6.2) is sought. Therefore, if for all $i \in \{1, 2, \cdots, L\}$

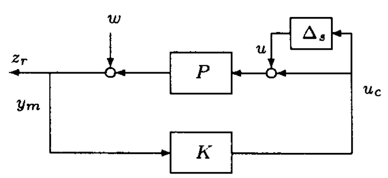
$$\begin{bmatrix} A_i^T P + P^T A_i + C_i^T Q C_i & P^T B_i + C_i^T Q D_i \\ B_i^T P + D_i^T Q C_i & D_i^T Q D_i - Q \end{bmatrix} \le 0.$$

have a common constant solution (P, Q). Then (P, Q) is also a solution to (6.2), the corresponding Lyapunov function is $V(x) = x^T P x$.

This treatment suggests a tractable algorithm to get local solutions, which can be used to seek constant solutions on each partitioned state set $\mathbf{N}_i$ in the proof of theorem 6.2. However, this approach generally leads to conservative results if the prescribed state set is large enough [12].

We end this paper by a simple example.

Example 6.4 Consider an uncertain feedback system with the following block diagram.



where P is the nonlinear plant; K is the controller such that the output z_r is supposed to be regulated; Δ_s is the causal uncertainty with $\mathcal{L}_2$-gain $\le \frac{1}{\sqrt{2}}$ satisfying Assumption 3.1. P has the following realization.

$$P : \begin{cases} \dot{x} = e^x(u + u_c) \\ z = x + w \\ y_m = x + w \end{cases}$$

To analyze the feedback system, we need to check that given $\gamma > 0$ and K, whether or not $\int_0^T \|z_r\|^2 \, dt \le \gamma^2 \int_0^T \|w\|^2 \, dt, \forall T \in \mathbf{R}^+$, for all admissible Δ_s. Let $\gamma = \frac{1}{\sqrt{2}}$, $K = -1$. To standardize the problem, define $\Delta := \sqrt{2}\Delta_s$, $y := \frac{1}{\sqrt{2}} u_s$, $z = \epsilon \frac{1}{\sqrt{2}} z_r$ for $\epsilon < 1$. We therefore have system G as follows,

$$G : \begin{cases} \dot{x} = -e^x x - e^x u + e^x w \\ y = -\frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} u \\ z = \epsilon \frac{1}{\sqrt{2}} x + \epsilon \frac{1}{\sqrt{2}} u \end{cases}$$

It is sufficient to check if the standard feedback system (G, Δ) has robust performance. The corresponding NLMI is as follows,

$$\begin{bmatrix} -2e^x P(x) + \frac{1}{2}(Q + \epsilon^2) & -e^x P(x) + \frac{1}{2}(Q + \epsilon^2) & e^x P(x) \\ -e^x P(x) + \frac{1}{2}(Q + \epsilon^2) & \frac{1}{2}(\epsilon^2 - Q) & 0 \\ e^x P(x) & 0 & -1 \end{bmatrix} < 0$$

There exist positive solutions $(Q, P(x))$ to the above two inequalities; they satisfy $Q = 1$ and $\frac{1+\epsilon^2}{3-\epsilon^2} e^{-x} < P(x) < e^{-x}$. Therefore, the $\mathcal{L}_2$-gain for the closed loop system $\le \frac{1}{\sqrt{2}\epsilon}$ for all $\epsilon < 1$, which in turn implies the $\mathcal{L}_2$-gain $\le \frac{1}{\sqrt{2}}$ for all admissible Δ_s.

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